

CLOSED SETS OF MAHLER MEASURES

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ABSTRACT. Given a k -variable Laurent polynomial F , any $\ell \times k$ integer matrix A naturally defines an ℓ -variable Laurent polynomial F_A . I prove that for fixed F the set $\mathcal{M}(F)$ of all the logarithmic Mahler measures $m(F_A)$ of F_A for all A is a closed subset of the real line. Moreover, the matrices A can be assumed to be of a special form, which I call *Primitive Hermite Normal Form*. Furthermore, if F has integer coefficients and $\mathcal{M}(F)$ contains 0, then 0 is an isolated point of this set.

I also show that, for a given bound $B > 0$, the set \mathcal{M}_B of all Mahler measures of integer polynomials in any number of variables and having length (sum of the moduli of its coefficients) at most B is closed. Again, 0 is an isolated point of \mathcal{M}_B .

These results constitute evidence consistent with a conjecture of Boyd from 1980 to the effect that the union \mathcal{L} of all sets \mathcal{M}_B for $B > 0$ is closed, with 0 an isolated point of \mathcal{L} .

1. INTRODUCTION

The Mahler measure $M(f)$ of a polynomial $f(z) = a \prod_i (z - \alpha_i) \in \mathbb{C}[z]$ is defined by $M(f) := |a| \prod_i \max(1, |\alpha_i|)$. It arose first in a paper of D.H. Lehmer [8], as a way of estimating the growth rate of integer sequences defined by a linear recurrence. (Lehmer was using such sequences to generate primes.) Much later, Kurt Mahler [11] used it to bound the discriminant of a polynomial from above, and the minimum spacing of the roots of a polynomial from below. As noted for instance in [9], $m(f) := \log M(f)$ has the integral representation via Jensen's Theorem as

$$m(f) = \int_0^1 \log |f(e^{2\pi it})| dt.$$

This formula has the advantage that it immediately suggests a generalisation of Mahler measure to polynomials in several variables. So, following Mahler [10], let $k \geq 1$, $\mathbf{z}_k = (z_1, \dots, z_k)$ and $F(\mathbf{z}_k)$ be a nonzero Laurent polynomial with complex coefficients. Then its (logarithmic) Mahler measure $m(F)$ is defined as

$$m(F) = \int_0^1 dt_1 \cdots \int_0^1 dt_k \log |F(e^{2\pi it_1}, \dots, e^{2\pi it_k})| \quad (1)$$

Its (classical) Mahler measure is then $M(F) = \exp(m(F))$; for our purposes here it is slightly more convenient to work with $m(F)$ rather than $M(F)$.

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Next, given $\ell \geq 0$ and an $\ell \times k$ matrix $A = (a_{ij}) \in \mathbb{Z}^{\ell \times k}$, define, following [14], the k -tuple \mathbf{z}_ℓ^A by

$$\mathbf{z}_\ell^A = (z_1, \dots, z_\ell)^A = (z_1^{a_{11}} \cdots z_\ell^{a_{\ell 1}}, \dots, z_1^{a_{1k}} \cdots z_\ell^{a_{\ell k}})$$

(which is $(1, 1, \dots, 1) \in \mathbb{Z}^k$ when $\ell = 0$) and $F_A(\mathbf{z}_\ell) = F(\mathbf{z}_\ell^A)$, a polynomial in ℓ variables z_1, \dots, z_ℓ . Then $m(F_A)$ is defined by (1) with $F = F_A$ and $k = \ell$. Denote by $\mathcal{P}(F)$ the set

$$\mathcal{P}(F) := \{F_A : A \in \mathbb{Z}^{\ell \times k}, \ell \geq 0, k \geq 1\},$$

and by $\mathcal{M}(F)$ the set

$$\mathcal{M}(F) := \{m(F_A) : F_A \in \mathcal{P}(F), F_A \neq 0\}.$$

In particular, taking $\ell = 1$, we write A as the row vector $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{Z}^k$, and so $F_{\mathbf{r}}(z) = F(z^{r_1}, \dots, z^{r_k})$. Denote by $\mathcal{M}_1(F)$ the set $\{m(F_{\mathbf{r}}) \mid \mathbf{r} \in \mathbb{Z}^k\}$, with $m(F_{\mathbf{r}})$ defined by (1) with $F = F_{\mathbf{r}}$ and $k = 1$.

Our first result is as follows.

Theorem 1. *Let F be a nonzero Laurent polynomial with complex coefficients. Then the set $\mathcal{M}(F)$ is a closed subset of \mathbb{R} . Moreover, it is the closure $\overline{\mathcal{M}_1(F)}$ of $\mathcal{M}_1(F)$ in \mathbb{R} .*

Thus not only are all the $m(F_A)$ in $\overline{\mathcal{M}_1(F)}$, but also they are the *only* elements of $\overline{\mathcal{M}_1(F)}$.

In Proposition 13 a lower bound for $\mathcal{M}(F)$ is given, which implies that $\mathcal{M}(F) \in [0, \infty)$ when F has integer coefficients. In this case all polynomials in $\mathcal{P}(F)$ also have integer coefficients, and we can state our next result.

Theorem 2. *Suppose that $k \geq 1$, that the nonzero Laurent polynomial $F(\mathbf{z}_k)$ has integer coefficients and that $0 \in \mathcal{M}(F)$. Then 0 is an isolated point of $\mathcal{M}(F)$.*

The case where $F(\mathbf{z}_k)$ is a linear form was proved in 1977 by Lawton [6].

When F has integer coefficients and $\mathcal{M}(F)$ consists of more than just the single element 0, Theorems 1 and 2 tell us that this set has a smallest positive element. I call this the *Lehmer element* of $\mathcal{M}(F)$, and denote it by $\ell(F)$. It is discussed in Section 8.

These results are consistent with a far-sighted conjecture of Boyd [2] to the effect that the set \mathcal{L} of Mahler measures of *all* polynomials with integer coefficients in any number of variables is a closed subset of \mathbb{R} . But of course it is some way from proving it! As Boyd pointed out, the truth of his conjecture implies easily that 0 is an isolated point of \mathcal{L} . It is clear that Boyd's set \mathcal{L} is a countable union of sets $\mathcal{M}(F)$. In fact \mathcal{L} can be written as a countable *nested* union of a sequence of such sets – see Proposition 14.

Theorems 1 and 2 can be used to prove another result in the direction of Boyd's conjecture.

Theorem 3. *Let $B > 0$ be given. The set \mathcal{M}_B of all Mahler measures of integer polynomials in any number of variables and having length (sum of the moduli of its coefficients) at most B is closed. Furthermore, 0 is an isolated point of the set.*

Actually, the fact that 0 is an isolated point of \mathcal{M}_B already follows from results of Mignotte [12] (see also Stewart[18] and the survey article [17, Section 5.4]), who gave the lower bound $2^{1/(2B)}$ for the Mahler measure of integer noncyclotomic one-variable polynomials of length at most B . On applying the fact, from Theorem 1, that $\mathcal{M}_1(F)$ is dense in $\mathcal{M}(F)$, we see that Mignotte's lower bound is valid for all positive elements of \mathcal{M}_B .

In another paper [5, Theorem 2], Dobrowolski and I have just proved a theorem similar to Theorem 3, but for the set of Mahler measures of integer polynomials that are sums of a bounded number of monomials.

Boyd's conjecture is a substantial generalisation of a question asked by Lehmer [8] in 1933 as to whether, in the set $\{m(F)\}$ of Mahler measures of all nonzero one-variable polynomials F with integer coefficients, the point 0 is isolated. As is well known, the smallest known such $m(F) > 0$ is $m(L(z)) = \log(1.1762808) = 0.1623576 \dots$, where

$$L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1, \quad (2)$$

first discovered by Lehmer himself [8]. In Section 9 I state a new conjecture, which implies Boyd's conjecture. Whether this conjecture turns out to be any more tractable than his will be interesting to see! In that section, I also state another conjecture which, if provable, would answer a question Boyd posed at the end of [2].

We can in fact severely restrict the matrices A that are needed to produce all the different elements of $\mathcal{M}(F)$ in Theorem 1. To do this we need to make the following definition. We say that a matrix $H \in \mathbb{Z}^{\ell \times k}$ is in *Primitive Hermite Normal Form* if H is in Hermite normal form and the intersection of the \mathbb{R} -vector space spanned by the rows of H with \mathbb{Z}^k is equal to the integer lattice spanned by the rows of H . For discussion on this form of an integer matrix, see Section 3 below.

Theorem 4. *We have*

$$\mathcal{M}(F) = \{m(F_H) : H \in \cup_{\ell=0}^k \mathbb{Z}^{\ell \times k}, H \text{ of rank } \ell \text{ in Primitive Hermite Normal Form}\}.$$

In particular, this result shows that elements $m(F_A)$ make no contribution to $\mathcal{M}(F)$ when A has more rows than columns. Further, by Proposition 17 below, the only $m(F_A)$ with A a square matrix that makes a contribution is the $k \times k$ identity matrix, giving $m(F_A) = m(F)$.

Using this result, we can describe $\mathcal{M}(F)$ explicitly for small k , for instance for $k = 1, 2, 3$. For $k = 1$ there are two types: $m(F(1))$ (corresponding to the empty matrix A), and $m(F(z))$, corresponding to the 1×1 matrix $A = (1)$. For $k = 2, 3$, see Tables 1 and 2.

From Theorem 4 it is clear that the elements of the set $\{m(F_A) : A \in \mathbb{Z}^{\ell \times k}\}$ given in Theorem 1 are not all different. But, furthermore, I am not even claiming that all the measures given in Theorem 4 are distinct. Indeed, Proposition 5, which follows, shows that $\overline{\mathcal{M}_F}$ can be $\{0\}$.

2. PRELIMINARIES

We now give some results needed for the proof of our Theorem.

rank(A)	Mahler measure	matrix A	range of exponents
0	$m(F(1, 1))$	empty matrix	$n > 0, \gcd(n, p) = 1$
1	$m(F(1, z_1))$	$(0 \ 1)$	
1	$m(F(z_1^n, z_1^p))$	$(n \ p)$	
2	$m(F(z_1, z_2))$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

TABLE 1. $\mathcal{M}(F)$ for $k = 2$.

rank(A)	Mahler measure	matrix A	ranges of exponents
0	$m(F(1, 1, 1))$	empty matrix	$n \in \mathbb{N}, p \in \mathbb{Z}, \gcd(n, p) = 1$ $n \in \mathbb{N}, p, q \in \mathbb{Z}, \gcd(n, p, q) = 1$
1	$m(F(1, 1, z_1))$	$(0 \ 0 \ 1)$	
1	$m(F(1, z_1^n, z_1^p))$	$(0 \ n \ p)$	
1	$m(F(z_1^n, z_1^p, z_1^q))$	$(n \ p \ q)$	
2	$m(F(1, z_1, z_2))$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$n \in \mathbb{N}, p \in \mathbb{Z}, \gcd(n, p) = 1$
2	$m(F(z_1^n, z_1^p, z_2))$	$\begin{pmatrix} n & p & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
2	$m(F(z_1^n, z_1^{n'} z_2^{p'}, z_1^{n''} z_2^{p''}))$	$\begin{pmatrix} n & n' & n'' \\ 0 & p' & p'' \end{pmatrix}$	
3	$m(F(z_1, z_2, z_3))$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$n, p \in \mathbb{N}, n', n'', p', p'' \in \mathbb{Z}, 0 \leq n' < p',$ $\gcd(n, p) = \gcd(n', p') = \gcd(p', p'') = 1$ $\gcd(n, n' - rp', n'' - rp'') = 1$ for $0 \leq r < n$

TABLE 2. $\mathcal{M}(F)$ for $k = 3$.

Proposition 5 (Boyd [1, Theorem 1] – see also Schinzel [14, [Section 3.4, Cor.17, p260] and Smyth [15, Cor. 1])). *Suppose that $F \in \mathbb{Z}[\mathbf{z}_k]$ for some $k \in \mathbb{N}$. Then $m(F) = 0$ if and only if F belongs to $\mathcal{P}(S)$ for some polynomial S of the form $S(\mathbf{z}_k) = \pm z_1 C_2(z_2) C_3(z_3) \cdots C_k(z_k)$ for some k , where C_2, \dots, C_k are cyclotomic polynomials.*

Lemma 6. *If $B \in \mathbb{Z}^{\ell' \times \ell}$ and $A \in \mathbb{Z}^{\ell \times k}$ then $(\mathbf{z}_\ell^B)^A = \mathbf{z}_\ell^{(BA)}$. Further, if $G = F_A \in \mathcal{P}(F)$ for some polynomial F , then $G_B = F_{BA}$.*

Proof. The first result comes from [14, Section 3.4, Cor. 1, p.223].

For the second result, we have

$$G_B(\mathbf{z}_{\ell'}) = (F_A)_B(\mathbf{z}_{\ell'}) = F_A(\mathbf{z}_\ell^B) = F((\mathbf{z}_\ell^B)^A) = F(\mathbf{z}_\ell^{BA}) = F_{BA}(\mathbf{z}_\ell),$$

as claimed. \square

Proposition 7. *For nonsingular $V \in \mathbb{Z}^{k \times k}$ we have $m(F(\mathbf{z}_k^V)) = m(F(\mathbf{z}_k))$ and $m(F_V) = m(F)$. Further, $m(F_{VA}) = m(F_A)$ for any $\ell \times k$ integer matrix A .*

Proof. For $m(F(\mathbf{z}_k^V)) = m(F(\mathbf{z}_k))$, see [15, Lemma 7]. (See also Schinzel [14, Section 3.4, Cor. 8, pp. 226] for the case $\det(V) = \pm 1$.) Then $m(F_V) = m(F)$ follows straight from the definition of F_V . Next, we have

$$m(F_{VA}(\mathbf{z}_\ell)) = m(F(\mathbf{z}_\ell^{VA})) = m(F((\mathbf{z}_\ell)^V)^A) = m(F_A(\mathbf{z}_\ell^V)) = m(F_A(\mathbf{z}_\ell)).$$

□

Lemma 8. *If $G \in \mathcal{P}(F)$ then $\mathcal{P}(G) \subseteq \mathcal{P}(F)$ and $\mathcal{M}(G) \subseteq \mathcal{M}(F)$.*

Proof. Suppose that $F = F(\mathbf{z}_k)$ and $G = G(\mathbf{z}_\ell)$. Then $G = F_A$ for some $A \in \mathbb{Z}^{\ell \times k}$, and for any $B \in \mathbb{Z}^{\ell' \times \ell}$ with $0 \leq \ell' \leq \ell$ we have $G_B = F_{BA}$ by Lemma 6. Note that $BA \in \mathbb{Z}^{\ell' \times k}$ with $0 \leq \ell' \leq \ell \leq k$. This proves the first assertion, from which the second assertion follows immediately. □

Lemma 9. *For any two multivariable Laurent polynomials F and G we have that $\mathcal{P}(FG) \subseteq \mathcal{P}(F)\mathcal{P}(G)$ and $\mathcal{M}(FG) \subseteq \mathcal{M}(F) + \mathcal{M}(G)$.*

Proof. For F, G polynomials in \mathbf{z}_k and $A \in \mathbb{Z}^{\ell \times k}$ and some ℓ with $0 \leq \ell \leq k$ we have

$$(FG)_A = F_A G_A \in \mathcal{P}(F)\mathcal{P}(G),$$

and hence $m((FG)_A) = m(F_A) + m(G_A)$. □

This immediately implies the following.

Corollary 10. *If $\mathcal{M}(G) = \{0\}$ (see Proposition 5), then $\mathcal{M}(FG) = \mathcal{M}(F)$.*

Next, given $\ell \geq 2$ and $\mathbf{r} = (r_1, \dots, r_\ell) \in \mathbb{Z}^\ell$, define, following Boyd [1, 2]

$$q(\mathbf{r}) = \min_{\mathbf{s} \in \mathbb{Z}^\ell} \{\max_{i=1}^\ell |s_i| : \mathbf{r} \cdot \mathbf{s} = 0\}.$$

Here $\mathbf{s} = (s_1, \dots, s_\ell)$. The function q measures, in some sense, how different in magnitude the r_i are.

Lemma 11 (Boyd [1, p. 118]). *Let $n \in \mathbb{N}$ and $\mathbf{r}_n = (1, n, n^2, \dots, n^{\ell-1})$. Then $q(\mathbf{r}_n) = n$ (and so goes to ∞ as $n \rightarrow \infty$).*

The next result was first conjectured by Boyd [1], who also proved in [2] some partial results in direction of his conjecture, including essentially the result for $k = 2$.

Proposition 12 (Lawton[7]). *Let $F(z_1, \dots, z_k)$ be a Laurent polynomial with complex coefficients, and suppose that $\mathbf{r}^{(1)}, \mathbf{r}^{(2)}, \dots, \mathbf{r}^{(n)}, \dots$ is a sequence of vectors in \mathbb{Z}^k with $q(\mathbf{r}^{(n)}) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\lim_{n \rightarrow \infty} m(F_{\mathbf{r}^{(n)}}) = m(F).$$

Earlier Boyd had proved this result when F does not vanish on \mathbb{T}^n , this being a special case of [1, Lemma 1], which states that for a continuous function $G : \mathbb{T}^k \rightarrow \mathbb{C}$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} G(z^{\mathbf{r}^{(n)}}) dz = \int_{\mathbb{T}^k} G(\mathbf{z}_k) d\mathbf{z}_k$$

for the same sequence of vectors $(\mathbf{r}^{(n)})$.

Proposition 13 (Smyth [15, Cor. 2]). *For a Laurent polynomial $F(\mathbf{z}) = \sum_{\mathbf{j} \in J} c(\mathbf{j}) \mathbf{z}^{\mathbf{j}} \in \mathbb{C}[z_1, \dots, z_k]$, where $J \subseteq \mathbb{Z}^k$, let the polytope $\mathcal{C}(F) \in \mathbb{R}^k$ be the convex hull of those $\mathbf{j} \in J$ with $c(\mathbf{j}) \neq 0$. Then*

$$\max_{\mathbf{j} \text{ an extreme point of } \mathcal{C}(F)} \log |c(\mathbf{j})| \leq m(F) \leq \log \left(\sum_{\mathbf{j} \in J} |c(\mathbf{j})| \right)$$

In particular, $m(F) \geq 0$ when F has integer coefficients.

The polytope $\mathcal{C}(F)$ is called the *exponent polytope* of F . [2, p. 460]. Let $\dim(F)$ denote its dimension, the *dimension* of F , which is clearly at most k .

Proposition 14. *The set \mathcal{L} can be written as a union $\cup_{n=1}^{\infty} \mathcal{M}(F^{(n)})$, where*

$$F^{(n)}(\mathbf{z}_{2n}) = z_1 + z_3 + \dots + z_{2n-1} - (z_2 + z_4 + \dots + z_{2n}).$$

Furthermore,

$$\mathcal{M}(F^{(1)}) \subseteq \mathcal{M}(F^{(2)}) \subseteq \mathcal{M}(F^{(3)}) \subseteq \dots \subseteq \mathcal{M}(F^{(n)}) \subseteq \dots$$

Proof. For a given $F(\mathbf{z}_k) = \sum_{\mathbf{j} \in J} c(\mathbf{j}) \mathbf{z}^{\mathbf{j}} \in \mathbb{Z}[z_1, \dots, z_k]$, we know that $\mathcal{M}((z_1 - 1)F(\mathbf{z}_k)) = \mathcal{M}(F(\mathbf{z}_k))$, by Corollary 10. So, replacing F by $(z_1 - 1)F$, if necessary, we can assume that $\sum_{\mathbf{j} \in J} c(\mathbf{j}) = F(1, \dots, 1) = 0$. Choose

$$n := \sum_{\mathbf{j} \in J \text{ with } c(\mathbf{j}) > 0} c(\mathbf{j}).$$

Then, for each \mathbf{j} with $c(\mathbf{j}) > 0$ replace $c(\mathbf{j})$ of the z_{2i-1} 's by $\mathbf{z}_k^{\mathbf{j}}$, and for each \mathbf{j} with $c(\mathbf{j}) < 0$ replace $(-c(\mathbf{j}))$ of the z_{2i} 's by $\mathbf{z}_k^{\mathbf{j}}$. This gives us the polynomial F in the form $F_A^{(n)}$, where A is a matrix where for each $\mathbf{j} \in J$ the matrix A has $|c(\mathbf{j})|$ of its columns equal to \mathbf{j} . Hence $m(F) \in \mathcal{M}(F^{(n)})$ for this value of n .

To show that the sequence of sets $(\mathcal{M}(F^{(n)}))$ are nested, it is enough observe that

$$F^{(n-1)}(\mathbf{z}_{2n-2}) = F^{(n)}(z_1, z_2, \dots, z_{2n-3}, z_{2n-2}, z_{2n-2}, z_{2n-2}),$$

so that $F^{(n-1)}$ is of the form $F_A^{(n)}$ for some (easily written down) matrix A , and hence that $F^{(n-1)} \in \mathcal{P}(F^{(n)})$. Applying Lemma 8, we see that $\mathcal{M}(F^{(n-1)}) \subseteq \mathcal{M}(F^{(n)})$, as claimed. \square

3. THE PRIMITIVE HERMITE NORMAL FORM OF AN INTEGER MATRIX

The canonical form for integer matrices which we now describe is a variant of the Hermite Normal Form.

Recall that a nonzero integer $\ell \times k$ matrix $A = (a_{ij})$ is in *Hermite Normal Form* (HNF) if it has the following properties:

- (1) For some integer r with $0 \leq r < k$ the leftmost r columns of A are zero;
- (2) For some integer s with $0 \leq s < \ell$ the bottom s rows of A are zero;

(3) For $1 \leq i \leq \ell - s$ there are integers j_i satisfying

$$r + 1 = j_1 < j_2 < \cdots < j_{\ell-s} \leq k$$

such that $a_{i,j_i} \in \mathbb{N}$, $a_{ij'} = 0$ for $j' < j_i$ and $0 \leq a_{i',j_i} < a_{i,j_i}$ for $i' < i$;

Note that A has rank $\ell - s$, and that $0 \leq \ell - s \leq k - r$. (The version of HNF described here is the row-echelon one.)

For *any* integer $\ell \times k$ matrix A there is some $U \in \text{GL}_\ell(\mathbb{Z})$ such that UA is in Hermite normal form – see [13, Ch. II, Section 6]. (Multiplying by U corresponds to applying a succession of row operations of the following kinds to A :

- ('Swap') Swap two rows;
- ('Add') Add an integer multiple of one row to a different row.)

This matrix $H = UA$ is then called the *Hermite Normal form* of A .

We now give a characterisation of matrices in *Primitive Hermite Normal Form* (PHNF) alternative to the definition given in the Introduction.

Proposition 15. *An integer $\ell \times k$ matrix A with rows $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_\ell$ is in PHNF if it is in HNF, and, additionally, for $i = 1, 2, \dots, \ell$ and every choice of integers u_{i+1}, \dots, u_ℓ the greatest common divisor (gcd) of all the components of $\underline{a}_i + \sum_{j=i+1}^\ell u_j \underline{a}_j$ is 1.*

Proof. First assume that H is as defined in the introduction. Then, for any real numbers $\lambda_1, \dots, \lambda_\ell$, if $\sum_{j=1}^\ell \lambda_j \underline{a}_j \in \mathbb{Z}^k$, it follows that the λ_j are all integers. But if $\underline{a}_i + \sum_{j=i+1}^\ell u_j \underline{a}_j$ has components with gcd = g for some integers u_{i+1}, \dots, u_ℓ then $\frac{1}{g} \underline{a}_i + \sum_{j=i+1}^\ell \frac{u_j}{g} \underline{a}_j \in \mathbb{Z}^k$. Therefore, by our assumption, g must be 1.

Conversely, assume A is in HNF and that for each $i = 1, 2, \dots, \ell$ and for each choice of integers u_i that the components of $\underline{a}_i + \sum_{j=i+1}^\ell u_j \underline{a}_j$ have gcd = 1. For any set of λ_j real, assume that the sum $\sum_{j=1}^\ell \lambda_j \underline{a}_j$ has integer components. From the fact that A is in row-echelon form, we see successively that $\lambda_1 \in \mathbb{Q}, \lambda_2 \in \mathbb{Q}, \dots, \lambda_\ell \in \mathbb{Q}$. Write $\lambda_j = n_j/g$, where the n_j are integers with gcd = 1, and g (the least common denominator of the λ_j 's) is a positive integer. Now suppose that $g > 1$, with say p a prime dividing g . Then the sum $\sum_{j=1}^\ell n_j \underline{a}_j$ has all components divisible by p . Suppose that h is the smallest index for which $p \nmid n_h$, and that $vn_h = 1 \pmod{p}$. Then the sum $\underline{a}_h + vn_{h+1} \underline{a}_{h+1} + \cdots + vn_\ell \underline{a}_\ell$ also has all its components divisible by p . But this contradicts our gcd = 1 assumption above. Hence $g = 1$, so that the λ_j are all integers. \square

If we allow ourselves to apply the following operation to A , additional to 'Swap' and 'Add' above:

- ('Scale') If a row has all entries divisible by $g > 1$, then divide that row by g ,

then we can reduce A to a matrix in PHNF. We call this matrix the *Primitive Hermite Normal Form* of A .

PHNF algorithm. First put $A \in \mathbb{Z}^{\ell \times k}$ into HNF – call it A again. Ignore any zero rows at the bottom of A , so that we can assume that A has rank $\ell \leq k$. We now start an ℓ -step process. The first step is to divide row ℓ by the gcd of its entries. Now add a

suitable multiple of (the new) row ℓ to row $\ell - 1$ so that the gcd of the entries of (the new) row $\ell - 1$ is as large as possible. Divide this row by this gcd. Now add suitable multiples of rows ℓ and row $\ell - 1$ to row $\ell - 2$ so that the gcd of the entries of (the new) row $\ell - 2$ is as large as possible. Divide this row by this gcd. Continue in this way. For the ℓ -th step, add suitable multiples of rows $\ell, \ell - 1, \dots, 2$ to row 1 so that the gcd of the entries of (the new) row 1 is as large as possible. Divide this row by this gcd. Finally, restore the resulting matrix to HNF by suitable row operations.

Incidentally, it should be clear from the row-echelon structure of A that the number of choices for sums of integer multiples of the lower rows to be added to the current row to give a gcd greater than 1 can be readily bounded.

Proposition 16. *For a matrix $A \in \mathbb{Z}^{\ell \times k}$, the above algorithm does indeed find its PHNF, $H \in \mathbb{Z}^{\ell \times k}$. Furthermore, we have $A = VH$ for some nonsingular $V \in \mathbb{Z}^{\ell \times \ell}$.*

Proof. Suppose that the algorithm does not find the PHNF H of A . Then, by Proposition 15, some row \underline{a}_j of H has the property that, for some integers u_j , $\underline{a}_i + \sum_{j=i+1}^{\ell} u_j \underline{a}_j$ has all its components divisible by some $g > 1$. But this readily leads to the conclusion that, in applying the algorithm to row i , the sums of multiples of the lower rows that were added did not give the largest possible gcd of the components, which it should have. Thus the algorithm works.

On applying the algorithm to A , each operation corresponds to either left multiplication of A by an element of $\text{GL}_{\ell}(\mathbb{Z})$ or left multiplication by D^{-1} , where D is an $\ell \times \ell$ diagonal matrix, all entries except one being 1, and the other entry being some integer $g > 1$. Hence indeed $A = VH$ for some nonsingular $V \in \mathbb{Z}^{\ell \times \ell}$. \square

Example. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 5 & 1 \end{pmatrix},$$

which is already in HNF. However, it is not in PHNF. To find its PHNF, add row 3 to row 2, and divide the modified row 2 by 2. Finally, subtract the twice-modified row 2 from row 1, to obtain

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 5 & 1 \end{pmatrix},$$

the PHNF of A .

The following result is an easy exercise in applying the PHNF algorithm.

Proposition 17. *If $A \in \mathbb{Z}^{k \times k}$ is nonsingular then its PHNF is the $k \times k$ identity matrix.*

The following result is needed for the proof of Theorem 4.

Proposition 18. *Suppose that $A \in \mathbb{Z}^{\ell' \times k}$ has rank $\ell \leq k$ and that its Primitive Hermite Normal Form is $H' = \begin{pmatrix} H \\ \mathbf{0} \end{pmatrix}$, where $H \in \mathbb{Z}^{\ell \times k}$ and $\mathbf{0}$ is the $(k - \ell) \times k$ zero matrix. Then $m(F_A) = m(F_H)$.*

Proof. We have that $A = VH'$ for some nonsingular $V \in \mathbb{Z}^{k \times k}$. Hence $m(F_A) = m(F_{H'})$ by Proposition 7. Furthermore, $F_{H'} = F_H$, by definition. \square

4. PROOF OF THEOREM 1

Proof. For the proof, we first show that every $m(F_A)$ lies in $\overline{\mathcal{M}_F}$. Then we show for any $\mathbf{r}^{(1)}, \mathbf{r}^{(2)}, \dots, \mathbf{r}^{(n)}, \dots$ in \mathbb{Z}^k with $m(F_{\mathbf{r}^{(n)}})$ converging, that its limit is of the form $m(F_A)$ for some A .

So, take any $A \in \mathbb{Z}^{\ell \times k}$, and let $\mathbf{r}^{(n)}$ be the sequence $(1, n, n^2, \dots, n^{k-1})$ from Lemma 11. Because $q(\mathbf{r}^{(n)}) \rightarrow \infty$ as $n \rightarrow \infty$ we can apply Proposition 12 to obtain

$$\lim_{n \rightarrow \infty} m(F_A(\mathbf{r}^{(n)})) = m(F_A).$$

Now for $\mathbf{r} = \mathbf{r}^{(n)}$ we have $F_A(\mathbf{r}) = F_{\mathbf{r}^A}$, so that $m(F_A)(\mathbf{r}) = m(F_{\mathbf{r}^A})$. Hence, for the sequence $\mathbf{r}^A = (\mathbf{r}^{(n)})^A \in \mathbb{Z}^k$ we have

$$\lim_{n \rightarrow \infty} m(F_{\mathbf{r}^A}) = m(F_A).$$

Hence $\{m(F_A) : A \in \mathbb{Z}^{k \times k}\} \subseteq \overline{\mathcal{M}_F}$.

To prove that these are the only limit points of \mathcal{M} , we take any sequence $\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(n)}, \dots \in \mathbb{Z}^k$ for which $m(F_{\mathbf{r}^{(n)}})$ converges. We separate the proof into three cases, doing the trivial case $k = 1$ first and then, for $k \geq 2$, separating the cases where the sequence $q(\mathbf{r}^{(n)})$ is either unbounded or bounded.

Case 1: $k = 1$. Here $F = F(z_1)$ and our sequence is $\{m(F(z^{r_1}))\}$, for some sequence of nonzero integers r_1 . But, applying Proposition 7 with $n = 1$ and $A = (r_1)$, we have that the sequence $\{m(F(z^{r_1}))\}$ is constant, each term being $m(F(z))$.

Case 2: $k \geq 2$ and $q(\mathbf{r}^{(n)})$ unbounded. Then there is a subsequence of the $\mathbf{r}^{(n)}$'s for which $\lim_{n \rightarrow \infty} q(\mathbf{r}^{(n)})$ tends to infinity on that subsequence. Thus, by replacing the sequence of the $\mathbf{r}^{(n)}$'s by that subsequence, we can assume that, as $n \rightarrow \infty$ both that $m(F_{\mathbf{r}^{(n)}})$ converges and that $q(\mathbf{r}^{(n)}) \rightarrow \infty$. Then we can apply Proposition 12 to conclude that $\lim_{n \rightarrow \infty} m(F_{\mathbf{r}^{(n)}}) = m(F)$.

Case 3: $k \geq 2$ and $q(\mathbf{r}^{(n)})$ bounded. Our proof is by induction. From Case 1, we already know that the result is true for $k = 1$. We now assume $k \geq 2$ and that the result is true for all Laurent polynomials F in fewer than k variables.

Take a convergent sequence of real numbers $m(F_{\mathbf{r}})$ for $\mathbf{r} = \mathbf{r}^{(n)}$ ($n = 1, 2, 3, \dots$) such that the integer sequence $(q(\mathbf{r}))$ is bounded. Then there are only finitely many possibilities for the nonzero vectors $\mathbf{s} \in \mathbb{Z}^k$ in the definition of q such that $\mathbf{s} \cdot \mathbf{r} = 0$. Hence, by the Pigeonhole Principle, we can find an infinite subsequence of integers n for which the

corresponding sequence of vectors \mathbf{s} is constant. On replacing our original sequence $n = 1, 2, 3, \dots$ by this subsequence, we can assume that *all* \mathbf{r} satisfy $\mathbf{r} \cdot \mathbf{s} = 0$.

Next, take a $(k-1) \times k$ integer matrix U whose rows are a basis of the sublattice $L_{\mathbf{s}} := \{\mathbf{r} \in \mathbb{Z}^k \mid \mathbf{r} \cdot \mathbf{s} = 0\}$ of \mathbb{Z}^k . Then each $\mathbf{r} \in L_{\mathbf{s}}$ can be written as $\mathbf{c}U$ for some $\mathbf{c} \in \mathbb{Z}^{k-1}$. Then writing $G(\mathbf{z}_{k-1}) := F_U(\mathbf{z}_{k-1})$, a Laurent polynomial in at most $k-1$ variables, we have from Lemma 6 that $G_{\mathbf{c}} = F_{\mathbf{c}U} = F_{\mathbf{r}}$. Hence, applying the induction hypothesis to G , or Case 2 if $k-1 \geq 2$ and the sequence $\{q(\mathbf{c})\}$ is unbounded, we see that the sequence $\{m(F_{\mathbf{r}}(z))\} = \{m(G_{\mathbf{c}}(z))\}$ has a limit of the form $m(G_B)$ for some $B \in \mathbb{Z}^{\ell \times (k-1)}$ and some $\ell \leq k-1$.

Next, we note that, by Lemma 6 again, $G_B = F_A$, where $A = BU \in \mathbb{Z}^{\ell \times k}$. Hence $\overline{\mathcal{M}_F} \subseteq \{m(F_A) : A \in \mathbb{Z}^{\ell \times k}\}$, and so $\overline{\mathcal{M}_F} = \{m(F_A) : A \in \mathbb{Z}^{\ell \times k}, \ell \geq 0\}$, as claimed. \square

5. PROOF OF THEOREM 2

Proof. Suppose that $0 \in \mathcal{M}(F)$, but that it is not isolated. Then, because this set is the closure of the set of measures $m(F_{\mathbf{r}})$ of polynomials $F_{\mathbf{r}}$ for $\mathbf{r} \in \mathbb{Z}^k$, we can take a sequence of such polynomials $F_{\mathbf{r}^{(n)}}$ such that none of the $m(F_{\mathbf{r}^{(n)}})$ are 0, but $\lim_{n \rightarrow \infty} m(F_{\mathbf{r}^{(n)}}) = 0$. However, by Proposition 12, this limit is $m(F)$, which is therefore 0.

As in the proof of Theorem 1, we now separate three cases.

Case 1: $k = 1$. Here $F = F(z_1)$ and our sequence is $\{m(F(z^{r_1}))\}$, for some sequence of nonzero integers r_1 . But, applying Proposition 7 with $n = 1$ and $A = (r_1)$, we have that the sequence $\{m(F(z^{r_1}))\}$ is constant, each term being $m(F(z))$. Hence $m(F) = 0$, and so all terms of the converging sequence are 0, a contradiction.

Case 2: $k \geq 2$ and $q(\mathbf{r}^{(n)})$ unbounded. Then, as in the proof of Case 2 of Theorem 7, there is a subsequence of the $\mathbf{r}^{(n)}$'s for which $\lim_{n \rightarrow \infty} q(\mathbf{r}^{(n)})$ tends to infinity on that subsequence. Thus, by replacing the sequence of the $\mathbf{r}^{(n)}$'s by that subsequence, we can assume that, as $n \rightarrow \infty$ both that $m(F_{\mathbf{r}^{(n)}}) \rightarrow 0$ and that $q(\mathbf{r}^{(n)}) \rightarrow \infty$.

Hence, by Proposition 5, F is of the form $\pm z$ times a product of cyclotomic polynomials $C(z)$, where each occurrence of the variable z is replaced by a (possibly different for each occurrence) monomial in z_1, \dots, z_k . Hence each $F_{\mathbf{r}}$ is of the form $\pm z$ times a product of cyclotomic polynomials $C(z)$, where each occurrence of the variable z is replaced by a (possibly different for each occurrence) power of z , assumed to be nonzero. So $m(F_{\mathbf{r}^{(n)}}) = 0$, contradicting the fact that these values are all assumed to be nonzero.

Case 3: $k \geq 2$ and $q(\mathbf{r}^{(n)})$ bounded. Here, we follow quite closely the induction argument in Case 3 of the proof of Theorem 1. Thus the result is true for $k = 1$ by Case 1, so we assume that $k \geq 2$ and that the result is true for all F in fewer than k variables. Following that argument, we get that our sequence $\{m(F_{\mathbf{r}^{(n)}})\}$ has limit $m(F_{A'})$, where $A' \in \mathbb{Z}^{\ell \times k}$ for some $\ell \leq k-1$. Thus $m(F_{A'}) = 0$, and so, again by Proposition 5, F is of the form $\pm z$ times a product of cyclotomic polynomials $C(z)$, where each occurrence of the variable z is replaced by a (possibly different for each occurrence) monomial in z_1, \dots, z_k .

From the definition of $F_{A'}$, we then see that F itself has the same property. So, as in Case 2, we conclude that $m(F_{\mathbf{r}^{(n)}}) = 0$ for all n , giving the same contradiction again. \square

6. PROOF OF THEOREM 3

Proof. Consider all signed partitions $\mathbf{c} = (c_1, c_2, \dots, c_t)$ of all positive integers $b \leq B$. So the c_i are nonzero integers with $|c_1|, \dots, |c_t|$ nondecreasing and $\sum_{i=1}^t |c_i| = b$. For each such partition define the linear form

$$F_{\mathbf{c}}(\mathbf{z}_t) = c_1 z_1 + \dots + c_t z_t.$$

If $F_{\mathbf{c}}$ is of length b , then all $(F_{\mathbf{c}})_A \in \mathcal{P}(F_{\mathbf{c}})$ are of length at most b (there could be cancellation). Then every polynomial F of length at most B belongs to $\mathcal{P}(F_{\mathbf{c}})$ for some such \mathbf{c} , and so the set \mathcal{M}_B is the union of all such sets $\mathcal{M}(F_{\mathbf{c}})$. Since this is a finite union, with all the component sets being closed with 0 being an isolated point of the set, it follows that \mathcal{M}_B inherits these two properties. \square

7. PROOF OF THEOREM 4

Proof. Obviously

$$\{m(F_A) : A \in \cup_{\ell=0}^k \mathbb{Z}^{\ell \times k}, A \text{ of rank } \ell \text{ in PHNF}\} \subseteq \{m(F_A) : A \in \mathbb{Z}^{\ell' \times k}, \ell' \geq 0\} = \mathcal{M}(F).$$

In the other direction, we have by Proposition 18 that $m(F_A) = m(F_H)$, where $H' = \begin{pmatrix} H \\ \mathbf{0} \end{pmatrix}$ is the PHNF of A , with $H \in \mathbb{Z}^{\ell \times k}$ of rank $\ell \leq \min(\ell', k)$. Hence

$$\mathcal{M}(F) = \{m(F_A) : A \in \mathbb{Z}^{\ell' \times k}\} \subseteq \{m(F_H) : H \in \cup_{\ell=0}^k \mathbb{Z}^{\ell \times k}, H \text{ of rank } \ell \text{ in PHNF}\}.$$

\square

8. THE LEHMER ELEMENT OF $\mathcal{M}(F)$

Now $F^{(1)} = z_1 - z_2 = z_2(z_1 z_2^{-1} - 1)$, so $\mathcal{M}(F^{(1)}) = \{0\}$ by Proposition 5, and $\ell(F^{(1)})$ is not defined. Next, $z^4 - z^3 - z^2 + 1 \in \mathcal{P}(F^{(2)})$, and

$$m(z^4 - z^3 - z^2 + 1) = m((z^3 - z - 1)(z - 1)) = m(z^3 - z - 1) = \log(1.3247 \dots),$$

so that $\ell(F^{(2)}) \leq m(z^3 - z - 1)$. But in fact $\ell(F^{(2)}) = m(z^3 - z - 1)$, by virtue of a result of Dobrowolski [4, Proposition 2], who in fact proved that $m(z^3 - z - 1)$ was the minimal Mahler measure of *all* integer noncyclotomic quadrinomials.

Also $z^{11} - z^9 - z^8 + z^3 + z^2 - 1 \in \mathcal{P}(F^{(3)})$, and

$$m(z^{11} - z^9 - z^8 + z^3 + z^2 - 1) = m(L(z)(z - 1)) = m(L(z)) = \log(1.1762808 \dots),$$

where Lehmer's polynomial $L(z)$ is given by (2). This shows that $\ell(F^{(2)}) \leq \log(1.176 \dots)$. Because the $\mathcal{M}(F^{(n)})$'s are nested, it follows that $\ell(F^{(n)}) \leq m(L(z))$ for all $n \geq 3$.

Of course, being closed and clearly bounded, $\mathcal{M}(F)$ has a maximal element as well; let us call it $\ell_{\max}(F)$. For instance, since $m(z^4 + z^2 - z - 1) = \log(1.75487766624669) = 2m(z^3 - z - 1)$, we have $\ell_{\max}(F^{(2)}) \geq 2\ell(F^{(2)})$.

9. QUESTIONS AND CONJECTURES

1. Which points of $\mathcal{M}(F)$ are true limit points? For such a limit point, $m(F_A)$ say, there is a sequence $(\mathbf{r}^{(n)})$ for which $m(F_{\mathbf{r}^{(n)}}) \rightarrow m(F_A)$ with all $m(F_{\mathbf{r}^{(n)}})$ distinct. This question has been considered by Boyd [2, Appendix 2] for the polynomial $F(z_1, z_2) = 1 + z_1 + z_2$. See also the discussion in [3, Section 6].
2. Given $k \geq 2$, does there exist a integer polynomial F in k variables and of dimension k — see Section 2 for the definition — for which the elements of the multiset $\{m(F_H) : H \in \cup_{\ell=0}^k \mathbb{Z}^{\ell \times k}, H \text{ of rank } \ell \text{ in primitive Hermite Normal Form}\}$ from Theorem 4 are all distinct?

I make the following conjectures concerning the set \mathcal{L} of Mahler measures $m(F)$, where F is an integer polynomial in any number of variables.

Conjecture 19. *In any sequence $(m(F_n))_{n \in \mathbb{N}, F_n \in \mathcal{L}}$ where only finitely many of the $m(F_n)$ belong to any one set $\mathcal{M}(F)$ ($F \in \mathcal{L}$) we have that $m(F_n) \rightarrow \infty$ as $n \rightarrow \infty$.*

If this conjecture is true, then any convergent sequence in \mathcal{L} must have an infinite convergent subsequence in some $m(F)$. As $\mathcal{M}(F)$ is closed, by Theorem 1, the sequence's limit is in $\mathcal{M}(F)$, and so is certainly in \mathcal{L} . This shows that Conjecture 19 implies Boyd's conjecture.

Conjecture 20. *Suppose that we have a sequence $(m(F_n))_{n \in \mathbb{N}}$ where the $F_n \in \mathcal{L}$ are irreducible and $\dim(F_n) \rightarrow \infty$ as $n \rightarrow \infty$. Then only finitely many of the $m(F_n)$ belong to any one set $\mathcal{M}(F)$ ($F \in \mathcal{L}$).*

If both conjectures are true, we see that any sequence $\{m(F_n)\}_{n \in \mathbb{N}}$ where the $F_n \in \mathcal{L}$ are irreducible and $\dim(F_n) \rightarrow \infty$ as $n \rightarrow \infty$ would have the property that $m(F_n) \rightarrow \infty$ as $n \rightarrow \infty$. This would answer affirmatively a question asked by Boyd [2, p. 461].

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